Dr. Kamlesh Kumar

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B.Sc. Part-1 (Hons), Paper – II

Leibnitz's theorem (Differential Calculus)

Statement:- If u and v are two functions of x, possessing derivatives of the nth order, then

$$(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + {}^nC_r u_{n-r} v_r + \cdots + {}^nC_n u v_n.$$

Proof:- Let $y = uv$, where u and v are functions of x.

Diff. w.r.t. x both sides, we get

$$y_1 = u_1 v + u v_1$$

$$\Rightarrow \quad y_1 = {}^1C_0 u_1 v + {}^1C_1 u v_1 \quad (\because {}^1C_0 = 1,\ {}^1C_1 = 1)$$

This shows Leibnitz's theorem is true for $n = 1$

Diff. y_1 w.r.t. x, we have

$$y_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + u v_2) = u_2 v + 2u_1 v_1 + u v_2$$

$$\Rightarrow \quad y_2 = {}^2C_0 u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2$$

$$[\because {}^2C_0 = 1,\ {}^2C_1 = 2,\ {}^2C_2 = 1]$$

This shows Leibnitz's theorem is true for $n = 2$

Similarly, we can show $n = 3, n = 4$, value is true.

Now, we suppose that Leibnitz's theorem is true for a particular value of n, say, m.

That is we suppose that

$$y_m = {}^mC_0 u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \cdots + {}^mC_{r-1} u_{m-r+1} v_{r-1} + {}^mC_r u_{m-r} v_r + \cdots + {}^mC_{m-1} u_1 v_{m-1} + {}^mC_m u v_m \quad \text{--- (1)}$$

Diff. both sides of eqn (1) w.r.t. x, we get

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$$y_{m+1} = {}^mC_0(u_{m+1}v + u_m v_1) + {}^mC_1(u_m v_1 + u_{m-1}v_2)$$
$$+ {}^mC_2(u_{m-1}v_2 + u_{m-2}v_3) + \cdots + {}^mC_{r-1}(u_{m-r+2}v_{r-1} + u_{m-r+1}v_r)$$
$$+ {}^mC_r(u_{m-r+1}v_r + u_{m-r}v_{r+1}) + \cdots$$
$$\cdots + {}^mC_{m-1}(u_2 v_{m-1} + u_1 v_m) + {}^mC_m(u_1 v_m + u v_{m+1})$$

$$\Rightarrow y_{m+1} = {}^mC_0 u_{m+1}v + ({}^mC_0 + {}^mC_1)u_m v_1 + ({}^mC_1 + {}^mC_2)u_{m-1}v_2$$
$$+ \cdots + ({}^mC_{r-1} + {}^mC_r)u_{m-r+1}v_r + \cdots$$
$$\cdots + ({}^mC_{m-1} + {}^mC_m)u_1 v_m + {}^mC_m u v_{m+1} \quad \text{——— ②}$$

we know that ${}^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$

Putting $r = 1, 2, 3, \ldots, m$ successively in this well-known result, we get

${}^mC_0 + {}^mC_1 = {}^{m+1}C_1$, ${}^mC_1 + {}^mC_2 = {}^{m+1}C_2, \ldots, {}^mC_{m-1} + {}^mC_m = {}^{m+1}C_m$

Also, we know that ${}^mC_0 = {}^{m+1}C_0 = 1$ and ${}^mC_m = {}^{m+1}C_{m+1} = 1$.

Substituting these values in eqn. ②, we get

$$y_{m+1} = {}^{m+1}C_0 u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1}v_2 \cdots$$
$$+ {}^{m+1}C_r u_{m+1-r}v_r + \cdots + {}^{m+1}C_m u_1 v_m + {}^{m+1}C_{m+1} u v_{m+1} \quad \text{——— ③}$$

Eqn. ③ shows that Leibnitz's theorem is true for $n = m+1$.
i.e. Leibnitz's theorem is true ~~of~~ for next higher value of n, i.e for $m+1$ also.

We already proved that this theorem is true for $n = 1, 2, \ldots$ and so on

Therefore, by mathematical induction, Leibnitz's theorem is true for every positive integer n,
i.e. we have

$$\boxed{(uv)_n = {}^nC_0 u_n v + {}^nC_1 u_{n-1}v_1 + {}^nC_2 u_{n-2}v_2 + \cdots + {}^nC_r u_{n-r}v_r + \cdots + {}^nC_n u v_n}$$